VC Dimension

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Contents

The VC-Dimension

Learnable Infinite-Size Classes

Theorem
Let $\mathcal{H} = \{\mathbb{1}_{[x<a]} : a \in \mathbb{R}\}$. Then $\mathcal{H}$ is PAC learnable, using the ERM rule, with sample complexity $m_{\mathcal{H}}(\epsilon, \delta) \leqslant \lceil \log(2/\delta)/\epsilon \rceil$.

Proof.

- Let $h^*(x) = \mathbb{1}_{[x<a=*]}$, with $L_D(h^*) = 0$.
- Let $a_0 < a^* < a_1$:

$$D_x(\{x \in (a_0, a^*)\}) = D_x(\{x \in a^*, a_1)\}) = \epsilon$$

- Given training data S, let $b_0 = \max\{x : (x, 1) \in S\}, b_1 = \min\{x : (x, 0) \in S\}$.
- Let $b_S \in (b_0, b_1)$ be an ERM hypothesis.
- $b_0 \geqslant a_0,\ b_1 \leqslant a_1 \implies L_D(h_S) \leqslant \epsilon$. Therefore,

$$D^m\left(L_D(h_S) > \epsilon\right) \leqslant D^m\left(b_0 < a_0 \vee b_1 > a_1\right)$$

- $D^m(b_0 < a_0) \leqslant e^{-\epsilon m}$, $D^m(b_1 > a_1) \leqslant e^{-\epsilon m}$.

Shattering

Definition (Restriction of H to C)
Let $\mathcal{H}$ be a class of functions from $\mathcal{X}$ to $\{0, 1\}$ and let $C = \{c_1, \ldots, c_m\} \subset \mathcal{X}$. The restriction of $\mathcal{H}$ to C is the set of functions from C to $\{0, 1\}$ that can be derived from $\mathcal{H}$. That is,

$$\mathcal{H}_C = \{(h(c_1), \ldots, h(c_m)) \ : \ h \in \mathcal{H}\},$$

where we represent each function from C to $\{0, 1\}$ as a vector in $\{0, 1\}^{|C|}$.

Definition (Shattering)
A hypothesis class $\mathcal{H}$ shatters a finite set $C \subset \mathcal{X}$ if the restriction of $\mathcal{H}$ to C is the set of all functions from C to $\{0, 1\}$. That is, $|\mathcal{H}_C| = 2^{|C|}$.

VC Dimension

Threshold Functions
Let $\mathcal{H}$ be the threshold functions over $\mathbb{R}$. $C = \{c_1\}$ is shattered by $\mathcal{H}$. A set $C = \{c_1, c_2\}, c_1 \leqslant c_2$ is no shattered by $\mathcal{H}$. Consider the labeling $(0, 1)$.

Definition (VC-Dimension)
The VC-dimension of a hypothesis class $\mathcal{H}$, is the maximal size of a set $C \subset \mathcal{X}$ that can be shattered by $\mathcal{H}$. If $\mathcal{H}$ can shatter sets of arbitrarily large size, then $\mathcal{H} = \infty$.

NFL Revisited

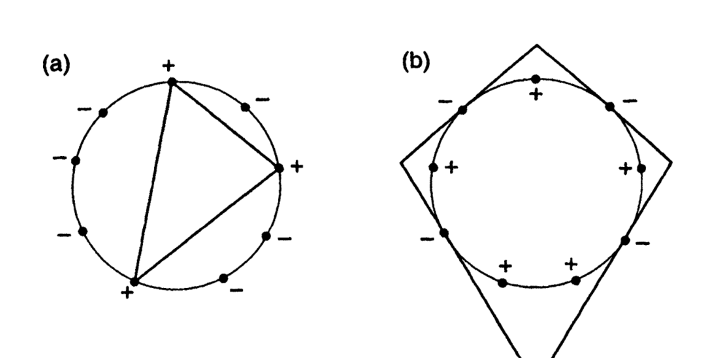
Let m be a training set size. If there exists a set $C \subset \mathcal{X}$ s.t. $|C| = 2m$ and $|\mathcal{H}_C| = 2^{2m}$,that is C is shattered by $\mathcal{H}$, then we cannot learn $\mathcal{H}$ with m samples.

Theorem
Let $\mathcal{H}$ be a class of infinite VC-dimension. Then, $\mathcal{H}$ is not PAC learnable.

Proof.
For any training set of m samples there exists a training set of $2m$ samples shattered by $\mathcal{H}$ $\rightarrow$ No Free Lunch. $\square$

Examples

- Threshold Functions on $\mathbb{R}$: 1.
- Intervals on $\mathbb{R}$: 2.
- Axis Aligned Rectangles on $\mathbb{R}^2$: 4.
- Hyperplanes in $\mathbb{R}^d$: $d+1$.
- Convex d-gons on $\mathbb{R}^2$: $2d+1$.
- Convex polygons on $\mathbb{R}^2$: ∞.
- $\{\sin(\theta x) : \theta \in \mathbb{R}\}$ on $\mathbb{R}$: ∞.



Growth Function

Definition (Growth Function)
Let $\mathcal{H}$ be a hypothesis class. Then the growth function of $\mathcal{H}$, denoted $\tau_{\mathcal{H}}(m) : \mathbb{N} \to \mathbb{N}$, is defined as

$$\tau_{\mathcal{H}}(m) = \max_{C \subset \mathcal{X} : |C| = m} |\mathcal{H}_C|.$$

Intervals
Let $\mathcal{H}$ be the class of intervals on $\mathbb{R}$, then

$$\tau_{\mathcal{H}}(1) = 2$$
$$\tau_{\mathcal{H}}(2) = 4$$
$$\tau_{\mathcal{H}}(3) = 7$$
$$\tau_{\mathcal{H}}(4) = 11$$

In general $\tau_{\mathcal{H}}(n) = (n+1)n/2 + 1 = O(n^2)$

Sauer's Lemma and UC

Lemma (Sauer's Lemma)
Let $\mathcal{H}$ *be a hypothesis class with* $\mathrm{VCdim}(\mathcal{H}) \leqslant d < \infty$. *Then, for all* m, $\tau_{\mathcal{H}}(m) \leqslant \sum_{i=0}^{d} \binom{m}{i}$. *In particular, if* $m > d+1$ *then* $\tau_{\mathcal{H}}(m) \leqslant (em/d)^d$.

Theorem (Generalized UC)
Let $\mathcal{H}$ *be a class and let* $\tau_{\mathcal{H}}$ *be its growth function. Then, for every* D *and ever* $\delta \in (0,1)$, *with probability of at least* $1-\delta$ *over the choice of* $S \sim D^m$ *we have*

$$|L_D(h) - L_S(h)| \leqslant \frac{4 + \sqrt{\log(\tau_{\mathcal{H}}(2m))}}{\delta\sqrt{2m}}$$

The Fundamental Theorem of Statistical Learning

Theorem
Let $\mathcal{H}$ be a hypothesis class of functions from a domain $\mathcal{X}$ to $\{0, 1\}$ and let the loss function be the $0 - 1$ loss. Then, the following are equivalent:

1. *$\mathcal{H}$ has the UC property.*
2. *Any ERM rule is a successful agnostic PAC learner for $\mathcal{H}$.*
3. *$\mathcal{H}$ is agnostic PAC learnable.*
4. *$\mathcal{H}$ is PAC learnable.*
5. *Any ERM rule is a successful PAC learner for $\mathcal{H}$.*
6. *$\mathcal{H}$ has a finite VC-dimension.*

Quantitative Version

Let $\mathcal{H}$ be a hypothesis class of functions from a domain $\mathcal{X}$ to $\{0, 1\}$ and let the loss function be the $0-1$ loss. Assume that $\mathrm{VCdim}(\mathcal{H}) = d < \infty$. Then, there are absolute constants C_1, C_2 such that:

1. $\mathcal{H}$ has the UC property/is APAC learnable with sample complexity:

$$C_1 \frac{d + \log(1/\delta)}{\epsilon^2} \leqslant m_{\mathcal{H}}^{UC} \leqslant C_2 \frac{d + \log(1/\delta)}{\epsilon^2}$$

2. $\mathcal{H}$ is PAC learnable with sample complexity:

$$C_1 \frac{d + \log(1/\delta)}{\epsilon} \leqslant m_{\mathcal{H}} \leqslant C_2 \frac{d + \log(1/\delta)}{\epsilon}$$

Constraint Sampling

Let $\mathcal{Z}$ be a feasible set of linear constraints

$$\gamma_z^T + k_z \geqslant 0, z \in \mathcal{Z}$$

where $K\ell^*|\mathcal{Z}|$.

Claim
The feasible region specified by all constraints can be closely approximated by a sampled subset.

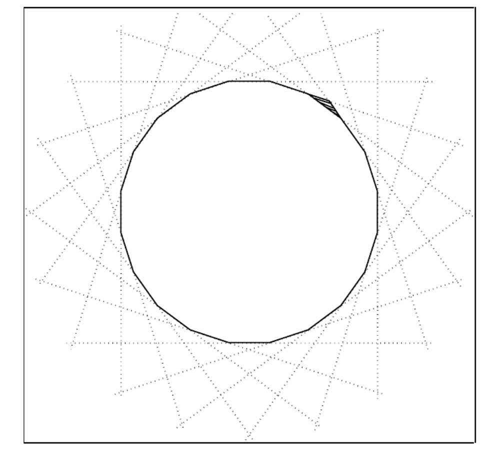
Let ψ be a probability measure over $\mathcal{Z}$. Given $\epsilon \in (0, 1)$ we want a $\mathcal{W} \subseteq \mathcal{Z}$ such that

$$\sup_{\{r \,:\, \gamma_z^T r + k_z \geqslant 0, z \in \mathcal{W}\}} \psi(\{y : \gamma_y^T r + k_y < 0\}) \leqslant \epsilon$$

$$\mathrm{VCdim}(\{\{(\gamma, k) : \gamma^T r + k \geqslant 0\} \;:\; r \in \mathbb{R}^K\}) = K$$

Constraint Sampling

Worst Case Constraint Set:



Questions?

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