Convex Optimization

Kontonis Vasilis

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Corelab, NTUA

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Convex Problems

Optimization Problem

General Optimization Problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leqslant 0, \; i = 1, \ldots, m \\ & h_i(x) = 0, \; i = 1, \ldots, p \end{aligned}$$

- Domain of the problem:

$$\mathcal{D} = \bigcap_{i=0}^{m} \text{dom} f_i \cap \bigcap_{i=1}^{p} \text{dom} h_i$$

- When a point x is feasible?
- $p^* = \inf\{f_0(x) \mid x \text{ is feasible}\}$
- A feasible point x with $f_0(x) \leqslant p^* + \epsilon$ is called ϵ-**suboptimal**.
- A feasible point x is locally optimal if it is optimal in a norm ball of radius $R > 0$.

Convex Optimization Problem (CP)

We minimize a **convex objective** over a **convex set**.

Generic Convex Problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i \leqslant 0, \; i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

- $f_0, \; f_i$ must be convex.
- The **equality constraints** must be affine.
- The feasible set of this problem is convex. Why?
- If f_0 is quasiconvex the problem is called quasiconvex.
- For convex and quasiconvex problems the optimal set, and the ϵ-suboptimal sets are convex.
- Locally optimal points are globally optimal.

Optimality Criteria

A point x is optimal iff $x \in X$ and

$$\nabla f_0(x)^T(y - x) \geqslant 0, \text{ for all } y \in X.$$

- Unconstraint problems $\nabla f_0(x) = 0$.
  Example: $f_0(x) = (1/2)x^T P x + q^T x + r$, $\nabla f_0(x) = x^T P + q^T$
- Equality Constraints

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & Ax = b \end{aligned}$$

$\nabla f_0(x) \perp \mathcal{N}(A) \implies \nabla f_0(x) \in \mathcal{R}(A^T)$. Therefore we obtain the Langrange Multiplier optimality condition:

$$\nabla f_0(x) + A^T \nu = 0$$

Quasiconvex Optimization

A quasiconvex problem can have locally optimal solutions that are **not** globally optimal.

Optimality Condition: A point x is optimal if

$$\nabla f_0(x)^T(y - x) > 0, \text{ for all } y \in X \setminus \{x\}.$$

Quasiconvexity of f_0 implies that there exist a family of convex functions $\phi_t : \mathbb{R}^n \to \mathbb{R},\ t \in \mathbb{R}$ such that $f_0(x) \leqslant t \Leftrightarrow \varphi_t(x) \leqslant 0$, moreover it holds $s \geqslant t \implies \varphi_s(x) \leqslant \varphi_t(x)$.

Consider the feasibility problem

$$\begin{aligned} \text{find} \quad & x \\ \text{subject to} \quad & \varphi_t(x) \leqslant 0 \\ & f_i(x) \leqslant 0,\ i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

Can we use this feasibility problem to derive an approximation algorithm for the Quasiconvex Optimization problem?

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Bisection

Linear Programming

Chebyshev Inequalities

Let X be a RV with values $\{u_1, \ldots, u_n\}$, $p_i = \Pr(X = u_i)$.
Assume that the distribution of X, namely the p_i, is unknown.
Assume that we have upper and lower bounds on expected values of some function of X and probabilities of some subsets of $\mathbb{R}$.

$$\mathbb{E}[f(X)] = \sum_{i=1}^{n} p_i f(u_i), \quad \Pr[x \in S] = \sum_{u_i \in S} p_i$$

LP:

$$\begin{array}{ll} \text{minimize} & a_0^T p \\ \text{subject to} & p \geqslant 0,\ 1^T p = 1, \\ & \alpha_i \leqslant a_i^T p \leqslant \beta_i,\ i = 1, \ldots, m \end{array}$$

Linear Fractional Programming

Suppose we want to minimize a ratio of linear functions $f_0(x) = \frac{c^T x + d}{e^T x + f}$, $\text{dom} f_0 = \{x \mid e^T x + f > 0\}$.

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & Gx \leqslant h \\ & Ax = b \end{aligned}$$

Equivalent **LP**:

$$\begin{aligned} \text{minimize (y,z)} \; & c^T y + dz \\ \text{subject to} \quad & Gy - hz \leqslant 0 \\ & Ay - bz = 0 \\ & e^T y + fz = 1 \\ & z \geqslant 0 \end{aligned}$$

To show equivalence consider the pair

$$y = \frac{x}{e^T x + f}, \quad z = \frac{1}{e^T x + f}$$

Questions?

References i

S. Boyd and L. Vandenberghe.
Convex Optimization.
Cambridge University Press, Cambridge, UK ; New York, Mar. 2004.